ONE METHOD OF SOLVING INCORRECTLY STATED PROBLEMS

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The possibility of obtaining a regular solution to incorrectly stated problems without determining the regularization parameter is discussed. Data are given on a calculation of a model example for an inverse boundary problem of heat conduction.

Many inverse problems of heat conduction are reduced to the solution of an equation of the first kind

$$Au = f_\delta \tag{1}$$

and are incorrectly stated problems. In [1-3] it was proposed that such problems be solved using the minimization of the smoothing functional

$$\Phi_\alpha(u_\alpha^\delta) = \inf_u \Phi_\alpha(u) = \inf_u \{\|Au - f_\delta\|^2_{L_2} + \alpha\|u'\|^2_{L_2}\}. \tag{2}$$

Then, with the appropriate choice of the regularization parameter α [$\alpha = O(\delta^2)$], the solution u_α^δ of the problem (2) will converge uniformly as $\delta \to 0$ to u_T, the solution of the problem (1) with exact data.

Let us consider the possibility of applying iteration methods of the steepest-descent type to the solution of the problem (2). We designate $L = d/dt$; $L: U \to L_2[0, \tau_m]$. Here $U \subset L_2[0, \tau_m]$ is a set of absolutely continuous functions whose derivatives are integrated with a square. It is assumed that $f_\delta \in L_2[0, \tau_m]$. Suppose that it is also known that $u_T \in U_0$, where

$$U_0 = \{u : u(0) = 0, \quad u \in U\}. \tag{3}$$

For the mapping $L: U_0 \to L_2[0, \tau_m]$. The continuous inverse mapping

$$Lv^{-1} = \int_0^\tau v(\xi)\, d\xi, \quad v \in L_2[0, \tau_m]$$

exists. For it the conjugate operator

$$L^{-1*}g = \int_\tau^{\tau_m} g(\xi)\, d\xi, \quad g \in L_2[0, \tau_m]$$

obviously exists. We introduce the operator $C = AL^{-1}$ into the analysis. Then the problem (2) is equivalent to the following variational problem:

$$\Phi_\lambda(v_\lambda^\delta) = \inf_{v \in L_2} \Phi(v) = \inf_{v \in L_2} \{\lambda\|Cv - f_\delta\|^2_{L_2} + \|v\|^2_{L_2}\}, \quad \lambda = \frac{1}{\alpha}. \tag{4}$$

If α is chosen from the discrepancy [4], then the problem (4) is none other than the finding of the conditional minimum of the functional $\Omega(v) = \|v\|^2_{L_2}$ in the set

$$D_\delta = \{v : \|Cv - f_\delta\|^2_{L_2} \leqslant \delta^2, \quad v \in L_2[0, \tau_m]\},$$

i.e., finding the projection of zero in the set D_δ.

To find the approximate solution to this problem we used the method of steepest descent with respect to the antigradient of the discrepancy. The iteration sequence for this method will have the form (for the case of a linear operator A)

$$v_{n+1} = v_n - \beta_n C^*(Cv_n - f_\delta), \quad v_0(\tau) = 0, \tag{5}$$

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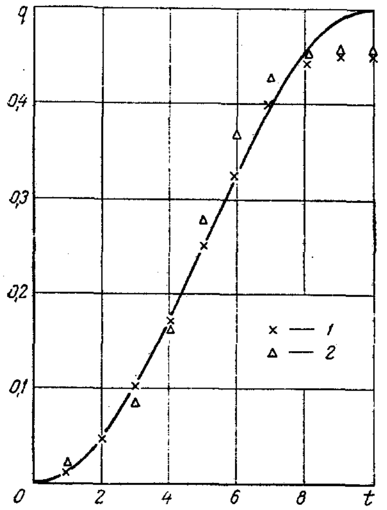

Fig. 1. Reconstruction of heat flux by the method of steepest descent: q) heat flux (10^6 W/m^2); t) time (sec); solid curve) exact solution; points: 1) flux reconstructed on exact data; 2) flux reconstructed on perturbed data (perturbation distributed by a uniform law with a maximum scatter equal to 10% of the maximum value of the exact data).

where

$$C^* = L^{-1^*} A^*; \quad \beta_n = \frac{\|C^*(Cv_n - f_\delta)\|^2}{\|CC^*(Cv_n - f_\delta)\|^2}.$$

By integrating Eq. (5) one can obtain the sequence directly in terms of the original problem:

$$u_{n+1} = u_n - \beta_n L^{-1} L^{-1^*} A^* (Au_n - f_\delta), \quad u_0(\tau) \equiv 0, \tag{6}$$

$$\beta_n = \frac{\|L^{-1^*} A^* (Au_n - f_\delta)\|^2}{\|AL^{-1} L^{-1^*} A^* (Au_n - f_\delta)\|^2}.$$

This procedure is realized on a computer rather easily.

The sequence (6) was obtained for a linear operator A. However, in the case of an operator A which is nonlinear but Frechet differentiable one can construct a sequence analogous to (5) and (6). For this one obviously must show that the discrepancy functional $J(v) = \|Cv - f_\delta\|^2$ will be Frechet differentiable in this case also, and one must find an expression for the gradient J(v).

Let B be a Frechet differentiable operator; i.e., $B(v + \Delta v)$ can be represented in the form

$$B(v + \Delta v) = Bv + B_v' \Delta v + \omega_B(v, \Delta v),$$

where B_V' is a linear operator, and

$$\lim_{\|\Delta v\| \to 0} \frac{\|\omega_B(v, \Delta v)\|}{\|\Delta v\|} = 0.$$

Then for $J(v) = \|Bv - f\|^2$ we have

$$J(v + \Delta v) - J(v) = \|B(v + \Delta v) - f\|^2 - \|Bv - f\|^2 = 2(Bv - f, B_v' \Delta v) +$$

$$+ 2((Bv - f), \omega_B(v, \Delta v)) + \|B_v' \Delta v + \omega_B(v, \Delta v)\|^2 = 2((B_v')^*(Bv - f), \Delta v) + \omega_1(v, \Delta v),$$

where it is obvious that

$$\lim_{\|\Delta v\|\to 0} \frac{|\omega_1(v, \Delta v)|}{\|\Delta v\|} = 0.$$

It remains to be shown that the differentiable operator C follows from the differentiable operator A. Actually,

$$C(v + \Delta v) - C(v) = A(L^{-1}v + L^{-1}\Delta v) - AL^{-1}v = A'_{L^{-1}v} L^{-1}\Delta v + \omega_A(L^{-1}v, L^{-1}\Delta v).$$

From the continuity and finiteness of the operator L^{-1} it follows that

$$\lim_{\|\Delta v\|\to 0} \frac{\|\omega_A(L^{-1}v; L^{-1}\Delta v)\|}{\|\Delta v\|} = \|L^{-1}\| \lim_{\|\Delta v\|\to 0} \frac{\|\omega_A(L^{-1}v; L^{-1}\Delta v)\|}{\|L^{-1}\|\cdot\|\Delta v\|} \leqslant \|L^{-1}\| \lim_{\|\Delta u\|\to 0} \frac{\|\omega_A(u, \Delta u)\|}{\|\Delta u\|} = 0.$$

Thus, the operator C is Frechet differentiable and its derivative has the form

$$C'_v = A'_{L^{-1}v} L^{-1}.$$

Then

$$(C'_v)^* = (L^{-1})^* (A'_{L^{-1}v})^*.$$

Hence, for the nonlinear operator A we have the iteration sequence

$$u_{n+1} = u_n - \beta_n L^{-1} L^{-1*} (A'_{u_n})^* (Au_n - f_\delta), \quad u_0(0) = 0.$$

Unfortunately, the equations for the choice of the step β_n in (5) and (6) are valid only for the linear case.

For inverse boundary problems of heat conduction the discrepancy gradient can be found from the solution of a problem conjugate to the problem in increments [5].

It should be noted that the requirement $u_T(0) = 0$ can be weakened in the linear case: It is sufficient to know the value of the reconstructed function at the left end of the interval of observation, and using the principle of superposition the problem is reduced to the case described above. The given method can be used for a problem in a nonlinear statement if $u_T(0)$ can be taken as equal to zero within the limits of accuracy of the solution of the problem.

An important feature of the given algorithm is the fact that the calculation of the regularization parameter is not required in it, although the solution of (4) is equivalent to the problem of the solution of (2) with the regularization parameter chosen with respect to the discepancy.

In accordance with this algorithm we solved a linear inverse problem of heat conduction consisting in the reconstruction of the heat flux at one of the boundaries of an infinite plane-parallel plate with a given temperature at the other boundary, which is thermally insulated. The results of the calculation of one of the model examples are presented in Fig. 1. The algorithm showed high stability perturbations of the initial information.

NOTATION

A, L, C, B	are the operators;
L^{-1}	is the operator inverse to operator L;
A^*	is the operator conjugate to operator A;
A_u^*	is the operator derivative of operator A at the point u;
u	is the element of space of solutions;
f_δ	is the initial information;
δ	is the error of initial information;
α	is the regularization parameter;
λ	is the Lagrange multiplier;
Φ_α, Φ_λ, J	are the functionals;
u_α^δ	is the point at which the functional $\Phi_\alpha(u)$ reaches the exact lower limit;
U	is the space of solutions;
U_0	is the subspace of the space U;
D_δ	is the set containining the unknown solution;
v	is the element of Hilbert space;

β_n is the depth of descent at n-th step along discrepancy gradient;
ω_1, ω_B, ω_A are the residual terms of equations for finite increments;
t, τ are the time;
τ_m is the right-hand value of complete time interval;
q is the heat flux.

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SOLUTION OF INVERSE COEFFICIENT PROBLEMS BY THE REGULARIZATION METHOD USING SPLINE FUNCTIONS

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The problem of determining the unknown coefficient in an equation of conservation of matter is discussed.

In a region $Q = \{(x, t): (0, 1) \times (0, 1)\}$ let the equation of conservation of matter be assigned in the form

$$\begin{gathered} Lu \equiv L_t u - aL_x^{(1)} u - \frac{da}{du} L_x^{(2)} u = f(x, t), \quad (x, t) \in Q, \\ u(x, 0) = \psi(x), \quad D_1 u = \varphi_1(t), \quad D_2 u = \varphi_2(t), \quad (x, t) \in \partial Q, \end{gathered} \tag{1}$$

where u(x, t) is the process under consideration; a(u) is an unknown coefficient; f(x, t) is a function of internal sources; $\psi(x)$, $\varphi_1(t)$, and $\varphi_2(t)$ are functions describing the initial and boundary conditions of the problem; L_t, $L_x^{(1)}$, and $L_x^{(2)}$ are differential operators expressing one or another conservation law; D_1 and D_2 are boundary-condition operators; ∂Q is the boundary of the region.

Within the framework of models with the simultaneous estimation of parameters the following formulations are known: first, when the coefficient is sought from an additional condition to the problem (1) [1, 2], and second, when it is sought from known δ-approximations to u and f, i.e., from elements $\hat{u}$ and $\hat{f}$ such that $\rho_U(u, \hat{u}) \le \delta_1$ and $\rho_F(f, \hat{f}) \le \delta_2$ [3]. The second formulation, although connected with a greater volume of measurements, still allows one to construct models of processes which are closer to the actual processes. We will have this formulation in mind below.

Following [4], we introduce the regularizing functional

$$\Phi_\alpha [a] = \iint\limits_Q (L\hat{u} - \hat{f})^2 \, dxdt + \alpha \Omega_{p,q}^{(k)}, \tag{2}$$

where α is the regularization parameter; $\Omega_{p,q}^{(k)}$ is a stabilizer of the form

$$\Omega_{p,q}^{(k)} = \begin{cases} \iint\limits_Q \left[p(a - a^*)^2 + \left(\dfrac{d^q a}{du^q} - \dfrac{d^q a^*}{du^q} \right)^2 \right] dxdt, & k = 1, \\ \iint\limits_Q \left[p \left(\dfrac{\partial^p a}{\partial x^p} - \dfrac{\partial^p a^*}{\partial x^p} \right)^2 + q \left(\dfrac{\partial^q a}{\partial t^q} - \dfrac{\partial^q a^*}{\partial t^q} \right)^2 \right] dxdt, & k = 2, \end{cases} \tag{3}$$

where a^* is the trial element.

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